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渦管の 3 次元不安定性再考

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概要

3 次元流中において 1 本の渦管に着目したとき，まわりの渦や背景流の影響は，まず移流として，次にその渦管の中心線上の各点を原点とする局所線形ひずみ流として近似的にとりこむことができよう．渦度に垂直な断面内に速度成分をもつ 2 次元純粋ずり流中におかれたまっすぐな渦管は 3 次元攪乱に対して線形不安定であることが，単純な設定のもとで Widnall 等によって具体的に明らかにされた [1)]．‘Widnall 不安定’ はそれ以後の渦管の安定性の議論における中心をなす：

局所純粋ずり流によって断面がわずかに楕円形に変形した渦柱を基本流にとる．断面が円形の渦は中立安定である．とくに，渦度分布が一様のものが ‘Rankine の渦’ で，その上に立つ無限個の固有振動モードは ‘Kelvin 波’ とよばれる．渦度ベクトルに垂直な断面 (xy-平面) 上に渦中心を原点とする極座標 (r,θ) を導入しよう．純粋ずり流は $\cos 2\theta$ と $\sin 2\theta$ によって特徴づけられる．いわゆる，‘四重極子’ 流である．この四重極子場を介して，Kelvin 波のうちの右巻き・左巻きの屈曲モード ($e^{i\theta}$, $e^{-i\theta}$ 成分) が共鳴を起こして指数関数的に成長する，というのがそのシナリオである．

本研究では，‘Widnall 不安定’ をハミルトン力学系におけるスペクトル理論の枠組みに乗せ，これを別の側面から眺めてみる．Krein の理論によると，単独の Kelvin モードに微小摂動を加えても不安定化することはなく，少なくとも 2 個のモードの波数 k と周波数 ω が一致して始めて不安定化が可能になる．すなわち，(k,ω) 平面上に描かれた分散関係を表す 2 本，もしくはそれ以上の，曲線が交差しなければならない．円形渦に対する摂動が四重極子流である場合には，$e^{im\theta}$, $e^{in\theta}$ 型の角度依存性をもつ 2 つの Kelvin モードが，関係 $|m-n|=2$ を満たすときに限って，分散曲線の交点上で共鳴を起こし得る．実際，2 つの屈曲モード $m=1$ と $n=-1$ はこの条件を満たしている．しかし，分散曲線が交差すればいつでも不安定化するというわけではない．攪乱がもつエネルギーの符号が問題で，同符号のエネルギーをもつ攪乱同士では不安化は起こり得ず，片方の攪乱のエネルギーが正で他方が負であることが共鳴不安定のための必要条件である．

さて，Widnall & Tsai (J. Fluid Mech. 1976) での固有値・固有関数の表式は不定積分を残したままになっており，当時の数値計算では十分な精度が得られなかったようである．実はこの積分は実行できて線形安定性問題の解が Bessel 関数を用いてあらわに書き下せることに気づいた．固有値・固有関数の高精度数値計算をもとに Widnall & Tsai (1976) の不備を修復し，さらに，‘Widnall 不安定’ と ‘楕円型不安定’ との関係について検討を加える (§4).

屈曲波の分散曲線の交点上で必ずパラメータ共鳴不安定が起こる．Kelvin 波のエネルギーの計算を実行し，この結果が Krein 理論と矛盾しないことを示す (§5).

これらの結果の要約については Fukumoto & Hattori (2002) も参照されたい．

1 Introduction

Moore & Saffman (1975) and Tsai & Widnall (1976), hereinafter being referred to as MS75 and TW76 respectively, addressed a three-dimensional linear stability problem of a straight vortex tube embedded in a plane pure shear flow with its principal axes perpendicular to the tube axis, and uncovered the essence of a parametric resonance between the left- and right-handed helical waves driven by the imposed shear. The *Moore-Saffman-Tsai-Widnall* instability is rather successfully adapted to an explanation for the short-wave instability of vortex rings (Widnall & Tsai 1977).

This paper makes an attempt at simplifying the results of TW76. We show that the Euler equations for the disturbance flow field are solvable, to first order in the shear strength ε, solely in terms of the Bessel and the modified Bessel functions, without having to include integrals. Compact formulae are available for the growth rate and the wavenumber range of unstable modes, whereby the numerical accuracy is considerably improved. Numerical computation leads us to a belief that the parametric resonance instability comes into play at every intersection point of bending waves. It is found that, in the short-wave limit, only the non-rotating modes survive; the maximum growth rate of rotating resonance instability modes decays to zero, whereas that of non-rotating modes approaches $9\varepsilon/16$, the value derived by Waleffe (1990).

A clue to an understanding of the wavenumber dependence of spectra lies in Krein's theory of parametric resonance in Hamiltonian systems (Krein 1950; MacKay 1986). A single Kelvin wave cannot be fed by a perturbation breaking the circular symmetry. The perturbation is the pure shear or a quadrupole field which is proportional to $\cos 2\theta$ and $\sin 2\theta$ in polar coordinates (r, θ). Subjected to a quadrupole field, two Kelvin waves of the form $\exp(im\theta)$ and $\exp(in\theta)$ with azimuthal wavenumber difference $|m - n| = 2$ can be cooperatively amplifiable when the eigenvalues $-i\omega_0$ collide on the imaginary axis as the parameter k_0 is varied, provided either that the collision eigenvalue is zero or that the signs of the wave energy are different from each other at a non-zero collision frequency. Since the wave energy is quadratic in amplitude, nonlinear solution of waves on the Rankine vortex would be wanted. Cairns' formula (Cairns 1979) offers an efficient machinery to bypass this procedure.

In §2, we give a concise description of Tsai-Widnall's formulation. Section 3 accommodates the linear dispersion relation. In §4, we integrate the linearized Euler equations to $O(\varepsilon)$ and examine how the quadrupole field affects the eigenvalues of the Kelvin waves. The eigenvalue problem to $O(\varepsilon)$ encounters a singularity at $\omega_0 = 0$. Care is exercised for these static waves in §4.2, which precedes a general case of $\omega_0 \neq 0$ in §4.3. In §5, the excess energy necessary to excite the Kelvin waves is calculated through Cairns' formula.

2 Setting of stability problem

In the absence of shear, the basic flow is a straight vortex tube with circular core of radius a and circulation Γ, surrounded by an irrotational flow of infinite expanse. Let us introduce cylindrical coordinates (r, θ, z) with the z-axis along the centerline of the core.

The radial coordinate r is normalized by the core radius a, the velocity by the maximum azimuthal velocity $\Gamma/2\pi a$, and the pressure by $\rho(\Gamma/2\pi a)^2$ with ρ being the density of fluid.

Suppose that the pure shear flow of strength ε is externally imposed. Let the r- and θ-components of velocity field be U and V, and the pressure be P. We denote by Φ the velocity potential for the irrotational flow field outside the core. The perturbation solution in powers of ε is represented, to $O(\varepsilon)$, as

$$\begin{aligned} U &= \varepsilon U_1(r,\theta) + \cdots, \quad V = V_0(r) + \varepsilon V_1(r,\theta) + \cdots, \\ P &= P_0(r) + \varepsilon P_1(r,\theta) + \cdots, \quad \Phi = \Phi_0(\theta) + \varepsilon \Phi_1(r,\theta) + \cdots. \end{aligned} \tag{2.1}$$

The subscript designates order in ε. The leading-order flow is the Rankine vortex:

$$V_0 = \begin{cases} r \\ 1/r, \end{cases} \qquad P_0 = \begin{cases} r^2/2 - 1 & (r \le 1) \\ -1/(2r^2) & (r > 1) \end{cases} \tag{2.2}$$

The first-order perturbation is a pure shear flow. Within the core,

$$U_1 = -r\sin 2\theta, \quad V_1 = -r\cos 2\theta, \quad P_1 = 0 \quad \text{for} \quad r < R(\theta,\varepsilon). \tag{2.3}$$

The velocity potential for the exterior flow is

$$\Phi_1 = (r^{-2} - r^2)\sin 2\theta/4. \tag{2.4}$$

The boundary shape $r = R(\theta,\varepsilon)$ of the core cross-section is an ellipse of small eccentricity with the major axis along the x-axis ($\theta = 0$):

$$R(\theta,\varepsilon) = 1 + \varepsilon\cos 2\theta/2 + O(\varepsilon^2). \tag{2.5}$$

We inquire into evolution of three-dimensional disturbances of infinitesimal amplitude superposed on this steady strained vortex. Suppose that the boundary of the core is disturbed to

$$r = R(\theta,\varepsilon) + \eta(\theta,z,t,\varepsilon), \tag{2.6}$$

where we assume superposition of normal modes for the disturbance amplitude η. Following the prescription of MS75 and TW76, we seek the solution for the disturbance velocity $\tilde{\boldsymbol{v}}$, the disturbance pressure $\tilde{p}$ and the external disturbance velocity-potential $\tilde{\phi}$ in a power series of ε to first order:

$$\tilde{\boldsymbol{v}} = (\boldsymbol{v}_0 + \varepsilon\boldsymbol{v}_1 + \cdots)e^{i(kz-\omega t)}, \quad \tilde{\phi} = (\phi_0 + \varepsilon\phi_1 + \cdots)e^{i(kz-\omega t)}. \tag{2.7}$$

Concomitantly, the wavenumber k and the frequency ω are also expanded as

$$k = k_0 + \varepsilon k_1 + \cdots, \quad \omega = \omega_0 + \varepsilon\omega_1 + \cdots. \tag{2.8}$$

3 Kelvin waves

Suppose that an infinitesimal-amplitude wave is superposed on the Rankine vortex, by deforming the circular core into

$$r = \eta(\theta, z, t) = 1 + A_0^{(m)} \exp[i(m\theta + k_0 z - \omega_0 t)]\,. \tag{3.9}$$

As a consequence, the disturbance flow field (2.7) takes, to $O(\varepsilon^0)$, the following form:

$$\boldsymbol{v}_0 = \boldsymbol{v}_0^{(m)}(r)e^{im\theta}\,, \quad \pi_0 = \pi_0^{(m)}(r)e^{im\theta}\,, \quad \phi_0 = \phi_0^{(m)}(r)e^{im\theta}\,. \tag{3.10}$$

With this form, the linearized Euler equations and the boundary conditions are cast into an eigenvalue problem. Define

$$\eta_m^2 = [4/(\omega_0 - m)^2 - 1]k_0^2\,, \tag{3.11}$$

and denote the z-component of disturbance velocity by w.

The bending waves ($m = \pm 1$) play the key role in the Moore-Saffman-Tsai-Widnall instability. Writing simply $\eta_\pm$ for $\eta_{\pm 1}$ and using the superscript $\pm$ for $m = \pm 1$ of the flow field, the eigenfunction and dispersion relation become

$$\phi_0^{(\pm)} = K_1(k_0 r)\alpha_0^{(\pm)} \qquad \text{for } r > \eta\,, \tag{3.12}$$

and

$$\begin{aligned}
\pi_0^{(\pm)} &= J_1(\eta_\pm r)\beta_0^{(\pm)}\,, \quad w_0^{(\pm)} = \frac{k_0}{\omega_0 \mp 1} J_1(\eta_\pm r)\beta_0^{(\pm)}\,,\\
u_0^{(\pm)} &= \frac{i}{2}\Big[-\Big(\frac{1}{\omega_0 \pm 1} + \frac{1}{\omega_0 \mp 3}\Big)\eta_\pm J_0(\eta_\pm r) + \frac{2}{\omega_0 \mp 3}\frac{J_1(\eta_\pm r)}{r}\Big]\beta_0^{(\pm)}\,,\\
v_0^{(\pm)} &= \pm\frac{1}{2}\Big[\Big(\frac{1}{\omega_0 \pm 1} - \frac{1}{\omega_0 \mp 3}\Big)\eta_\pm J_0(\eta_\pm r) + \frac{2}{\omega_0 \mp 3}\frac{J_1(\eta_\pm r)}{r}\Big]\beta_0^{(\pm)} \quad \text{for } r < \eta\,,
\end{aligned} \tag{3.13}$$

where

$$\alpha_0^{(\pm)} = -\frac{iJ_1(\eta_\pm)}{(\omega_0 \mp 1)K_1(k_0)}\beta_0^{(\pm)}\,, \tag{3.14}$$

and

$$\eta_\pm J_1(\eta_\pm)K_0(k_0) - k_0 J_0(\eta_\pm)K_1(k_0) \pm \frac{2(\omega_0 \pm 1)k_0}{(\omega_0 \mp 1)^2\eta_\pm} J_1(\eta_\pm)K_1(k_0) = 0\,. \tag{3.15}$$

4 Effect of pure shear

4.1 The equations for disturbances

We explore how the pure shear (2.3) of $O(\varepsilon)$ modifies Kelvin's dispersion relation. Given both types of bending waves to $O(\varepsilon^0)$,

$$\boldsymbol{v}_0 = \boldsymbol{v}_0^{(+)}e^{i\theta} + \boldsymbol{v}_0^{(-)}e^{-i\theta}\,, \tag{4.1}$$

then the wave excited to $O(\varepsilon)$ possess the following angular dependence:

$$\boldsymbol{v}_1 = \boldsymbol{v}_1^{(+)} e^{i\theta} + \boldsymbol{v}_1^{(-)} e^{-i\theta} + \boldsymbol{v}_1^{(0)} + \boldsymbol{v}_1^{(3)} e^{3i\theta} + \boldsymbol{v}_1^{(-3)} e^{-3i\theta}, \tag{4.2}$$

The general solution for the potential for the exterior velocity is readily obtainable as

$$\phi_1^{(\pm)} = K_1(k_0 r)\alpha_1^{(\pm)} - k_1 r K_0(k_0 r)\alpha_0^{(\pm)}, \tag{4.3}$$

where $\alpha_1^{(\pm)}$ is a constant imparted to the homogeneous part of solution.

The Euler equations for the vortical disturbance in the core is reduced to a second-order ordinary differential equation for the disturbance pressure $\pi_1^{(\pm)}$ in the following way:

$$\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} + \eta_\pm^2\right]\pi_1^{(\pm)} = \left\{\frac{8\omega_1}{(\omega_0 \mp 1)^3}k_0^2 - 2\frac{k_1}{k_0}\eta_\pm^2\right\}J_1(\eta_\pm r)\beta_0^{(\pm)}$$
$$+k_0^2\left\{\pm\left[\frac{1}{(\omega_0+1)^2} - \frac{1}{(\omega_0-1)^2}\right]\eta_\mp r J_0(\eta_\mp r) - \frac{3 \pm 2\omega_0}{(\omega_0 \pm 1)^2}J_1(\eta_\mp r)\right\}\beta_0^{(\mp)}. \tag{4.4}$$

The boundary conditions to be imposed at the core interface ($r = 1$) read, to $O(\varepsilon)$,

$$\frac{d\phi_1^{(\pm)}}{dr} - u_1^{(\pm)} = \frac{1}{4}\frac{du_0^{(\mp)}}{dr} \mp \frac{i}{2}v_0^{(\mp)} + \frac{1}{2}\phi_0^{(\mp)} - \frac{1}{4}\frac{d^2\phi_0^{(\mp)}}{dr^2}, \tag{4.5}$$

$$\pi_1^{(\pm)} - i(\omega_0 \mp 1)\phi_1^{(\pm)} = i\omega_1\phi_0^{(\pm)} \mp \frac{i}{2}\phi_0^{(\mp)} + \frac{i}{4}(\omega_0 \mp 1)\frac{d\phi_0^{(\mp)}}{dr} - \frac{1}{4}\frac{d\pi_0^{(\mp)}}{dr}. \tag{4.6}$$

So far we have kept track of the formulation of TW76. We are now in a position to show that (4.4) is explicitly solvable in a compact form. It will turn out that the solution suffers from a singularity at $\omega_0 = 0$ (see (4.17)), which calls for an individual treatment. We begin with the case of $\omega_0 = 0$ in §4.2, and the case of $\omega_0 \neq 0$ follows in §4.3.

4.2 The case of $\omega_0 = 0$

When restricted to $\omega_0 = 0$, (3.11) becomes

$$\eta_+ = \eta_- = \sqrt{3}k_0. \tag{4.7}$$

Inspection tells us that a general solution of (4.4) finite at $r = 0$ is

$$\pi_1^{(\pm)} = J_1(\sqrt{3}k_0 r)\beta_1^{(\pm)} + \left\{\pm\frac{4\omega_1}{\sqrt{3}}k_0 + \sqrt{3}k_1\right\}rJ_0(\sqrt{3}k_0 r)\beta_0^{(\pm)} + \frac{\sqrt{3}}{2}k_0 r J_0(\sqrt{3}k_0 r)\beta_0^{(\mp)}, \tag{4.8}$$

where $\beta_1^{(\pm)}$ is a constant associated with the homogeneous solution. The disturbance radial velocity field in the core is deduced at once by returning to the Euler equations as

$$u_1^{(\pm)} = \mp\frac{i}{3}\left\{\sqrt{3}k_0 J_0(\sqrt{3}k_0 r) + \frac{J_1(\sqrt{3}k_0 r)}{r}\right\}\beta_1^{(\pm)}$$
$$+i\left\{\omega_1\left[-\frac{7k_0}{3\sqrt{3}}J_0(\sqrt{3}k_0 r) + \left(\frac{4k_0^2}{3}r - \frac{1}{9r}\right)J_1(\sqrt{3}k_0 r)\right]\right.$$
$$\left.\mp k_1\left[\sqrt{3}J_0(\sqrt{3}k_0 r) - k_0 r J_1(\sqrt{3}k_0 r)\right]\right\}\beta_0^{(\pm)} \pm ik_0^2 r J_1(\sqrt{3}k_0 r)\beta_0^{(\mp)}, \tag{4.9}$$

Substitution of (4.3), (4.8) and (4.9), along with the solution (3.12) and (3.13) for the Kelvin waves, into the boundary conditions (4.5) and (4.6) with $\omega_0 = 0$, yields a coupled system of linear algebraic equations for $\alpha_1^{(\pm)}$ and $\beta_1^{(\pm)}$. This is further simplified with the aid of the dispersion relation (3.15). Hereinafter we use the shorthand notation $K_0 = K_0(k_0)$ and $K_1 = K_1(k_0)$. The resulting equations are then collected in matrix form as

$$\begin{pmatrix} -(k_0K_0 + K_1) & \pm\frac{i}{3}\left[\sqrt{3}k_0J_0(\sqrt{3}k_0) + J_1(\sqrt{3}k_0)\right] \\ \pm iK_1 & J_1(\sqrt{3}k_0) \end{pmatrix} \begin{pmatrix} \alpha_1^{(\pm)} \\ \beta_1^{(\pm)} \end{pmatrix} = \begin{pmatrix} F^{(\pm)} \\ G^{(\pm)} \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned} F^{(\pm)} = iJ_1(\sqrt{3}k_0)\Bigg\{&\left[\frac{\omega_1}{3}\Big(4k_0^2 - 5 - \frac{7k_0K_0}{K_1}\Big) \mp \frac{2k_1}{k_0}\Big(1 + \frac{k_0K_0}{K_1}\Big)\right]\beta_0^{(\pm)} \\ &\pm\Big(k_0^2 - \frac{1}{2} - \frac{k_0K_0}{2K_1}\Big)\beta_0^{(\mp)}\Bigg\}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} G^{(\pm)} = J_1(\sqrt{3}k_0)\Bigg\{&\left[\mp\omega_1\Big(\frac{11}{3} + \frac{4k_0K_0}{K_1}\Big) - \frac{2k_1}{k_0}\Big(1 + \frac{2k_0K_0}{K_1}\Big)\right]\beta_0^{(\pm)} \\ &-\Big(\frac{3}{2} + \frac{2k_0K_0}{K_1}\Big)\beta_0^{(\mp)}\Bigg\}. \end{aligned} \tag{4.12}$$

As is common, the matrix in (4.10) is identical with that to $O(\varepsilon^0)$. This solvability condition supplies ω_1 as a function of k_0.

The non-real ω_1 is permissible only when (??) for $m = \pm 1$ simultaneously posses a nontrivial solution $(\beta_0^{(+)}, \beta_0^{(-)}) \neq \mathbf{0}$, being indicative of parametric resonance (MS75; TW76). The instability occurs around the intersection point $k_1 = 0$ of the dispersion curves with $m = 1$ and -1. The maximum $\varepsilon\sigma_{1max}$ of the growth rate, attained at $_1 = 0$, and the half width $\varepsilon\Delta k_1$ of the unstable wavenumber range takes compact form:

$$\sigma_{1max} = \frac{3}{2}\left\{2\left(\frac{k_0K_0}{K_1}\right)^2 + \frac{3k_0K_0}{K_1} + k_0^2 + 1\right\}\Bigg/\left\{6\left(\frac{k_0K_0}{K_1}\right)^2 + \frac{8k_0K_0}{K_1} + 2k_0^2 + 3\right\}, \tag{4.13}$$

$$\Delta k_1 = \frac{K_1}{4K_0}\left\{2\left(\frac{k_0K_0}{K_1}\right)^2 + \frac{3k_0K_0}{K_1} + k_0^2 + 1\right\}\Bigg/\left(\frac{k_0K_0}{K_1} + 1\right). \tag{4.14}$$

We list in Table 1 the numerical values of σ_{1max} and Δk_1 for a first few intersection points with $\omega_0 = 0$.

In the short-wavelength regime ($k \gg 1$), (4.13) and (4.14) become

$$\sigma_{1max} = \frac{9}{16}\left(1 + \frac{1}{12k_0} - \frac{7}{48k_0^2} + \frac{5}{64k_0^3} + \cdots\right), \tag{4.15}$$

$$\Delta k_1 = \frac{3k_0}{4}\left(1 + \frac{1}{3k_0} + \frac{5}{24k_0^3} + \cdots\right). \tag{4.16}$$

k_0	σ_{1max}	Δk_1
0	0.5	∞
2.504982369	0.5707533917	2.145502816
4.349076726	0.5694562098	3.518286549
6.174012330	0.5681222780	4.883945142
7.993536550	0.5671646287	6.247280752

Table 1: The maximum growth rate $\varepsilon\sigma_{1max}$ and the half-width $\varepsilon\Delta k_1$ of unstable band to $O(\varepsilon)$. The case of $\omega_0 = 0$

The leading term of (4.15) is the well known value of the growth rate for the elliptical instability elaborated by Waleffe (1990). Roughly speaking, the Moore-Saffman-Tsai-Widnall instability for the static waves comprises the long-wave displacement instability and the short-wave instability. The latter is akin to and gives way, in the short-wave limit, to the elliptical instability.

4.3 The case of $\omega_0 \neq 0$

The general case of $\omega_0 \neq 0$ is dealt with in parallel with the case of $\omega_0 = 0$. Likewise, a general solution of (4.4) finite at $r = 0$ is expressible solely in terms of the Bessel functions:

$$\pi_1^{(\pm)} = J_1(\eta_\pm r)\beta_1^{(\pm)} - \left\{\frac{4\omega_1 k_0^2}{(\omega_0 \mp 1)^3 \eta_\pm} - k_1 \frac{\eta_\pm}{k_0}\right\} r J_0(\eta_\pm r)\beta_0^{(\pm)}$$
$$-\left\{\frac{1}{4}\eta_\mp r J_0(\eta_\mp r) \mp \frac{(\omega_0 \mp 1)^2(\omega_0+3)(\omega_0-3)}{32\omega_0} J_1(\eta_\mp r)\right\}\beta_0^{(\mp)}, \tag{4.17}$$

where $\beta_1^{(\pm)}$ is a constant. As anticipated at the end of §4.1, (4.17) diverges when $\omega_0 = 0$. In §4.2, this singularity was cured by virtually making the coefficient $\beta_1^{(\pm)}$ of the homogeneous part in (4.17) infinite so as to cancel this infinity.

Repeating the same procedure, we eventually obtain the expressions of σ_{1max} and Δk_1. The detail is reported elsewhere as they are lengthy. They are evaluated numerically at many of intersection points, showing $|\mathrm{Im}\omega_1| > 0$ with no exception. Table 2 accommodates the numerical results for a few crossing points that occur along the branch of the second radial mode and off the axis of $\omega_0 = 0$ but close to it.

Only the first row ($k_0 \approx 1.263926$) agrees with the numerics of TW76 up to the first two digits, but for the others, no digit of numerical values coincides with the corresponding one of TW76. Comparing with Table 1, we confirm the exclusive prevalence of non-rotating modes.

5 Energetics

X We are reminded of the point that the spectral stability of a Hamiltonian system can be lost only by eigenvalue collisions of positive- and negative-energy waves or by collisions

k_0	ω_0	σ_{1max}	Δk_1
1.263926	0.4060947	0.004071828	0.01193924
1.750370	0.5281003	0.004551168	0.01962740
2.113708	0.5994920	0.003797862	0.02140335
3.300462	0.1865562	0.007335005	0.03664937
5.173718	0.1238677	0.006114894	0.04520834
7.016673	0.0929365	0.005050467	0.04952554

Table 2: The maximum growth rate $\varepsilon\sigma_{1max}$ and the half-width $\varepsilon\Delta k_1$ of unstable band to $O(\varepsilon)$. The case of $\omega_0 \neq 0$

of eigenvalues at 0 (Krein 1950; MacKay 1986). Thus we are tempted to evaluate the energy of the Kelvin waves.

Cairns (1979) invoked an analogy from plasma physics and devised a trick for calculating wave energy that dispenses with a detailed knowledge of the global field. For the moment, we switch off the pure shear, but deal with the whole family of the Kelvin waves. A fundamental ingredient is the pressure on both edges of the vortex core $r = \eta(\theta, z, t)$ as given by (3.9). Let the augmented pressure on the vortex, through the interface $r = \eta$, acted by the surrounding fluid be $p_>(z, \theta, t)$, and the pressure on the surrounding fluid acted by the internal fluid be $p_<(z, \theta, t)$, and pose the following form:

$$p_>(\theta, z, t) = D_> A_0^{(m)} e^{i(m\theta + k_0 z - \omega_0 t)}, \quad p_<(\theta, z, t) = D_< A_0^{(m)} e^{i(m\theta + k_0 z - \omega_0 t)}. \tag{5.1}$$

Set the difference of the coefficients as

$$D(k_0, \omega_0) = D_>(k_0, \omega_0) - D_<(k_0, \omega_0). \tag{5.2}$$

The requirement of continuity $D = 0$ of pressure across the interface is no other than the dispersion relation. Cairns' formula prescribes the wave energy $E^{(m)}$, per unit length in z, to be

$$E^{(m)} = -\frac{\pi}{2}\omega_0 \frac{\partial D}{\partial \omega_0}(A_0^{(m)})^2. \tag{5.3}$$

We recall the pressure P_0 of the Rankine vortex and the disturbance pressure $\pi_0^{(m)}$ inside the core. Putting these together and using the dispersion relation, we can evaluate $P_0 + \pi_0^{(m)} \exp[i(m\theta + k_0 z - \omega_0 t)]$ at $r = \eta - 0$ to first order in wave amplitude $|A_0^{(m)}|$ complying with the form (5.1), with its coefficient provided by

$$D_< = 1 - \frac{(\omega_0 - m)^2(\eta_m/k_0)^2 J_{|m|}(\eta_m)}{\eta_m J_{|m|-1}(\eta_m) - \left(|m| + \dfrac{2m}{\omega_0 - m}\right) J_{|m|}(\eta_m)}. \tag{5.4}$$

The disturbance pressure $\pi_0^{(m)}$ in the exterior region is constructed from the disturbance velocity potential $\phi_0^{(m)}$ through $\pi_0^{(m)} = i\omega_0\phi_0^{(m)} - im\phi_0^{(m)}/r^2$. We evaluate the boundary value of the augmented pressure at $r = \eta + 0$ as above:

$$D_> = 1 - \frac{(\omega_0 - m)^2 K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}}. \tag{5.5}$$

Difference (5.2) between (5.5) and (5.4) is

$$D = (\omega_0 - m)^2 \left\{ \frac{(\eta_m/k_0)^2 J_{|m|}(\eta_m)}{\eta_m J_{|m|-1}(\eta_m) - \left(|m| + \dfrac{2m}{\omega_0 - m}\right) J_{|m|}(\eta_m)} - \frac{K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}} \right\}. \tag{5.6}$$

Provided that $\omega_0 \neq m$ as dictated in (??), the condition $D = 0$ indeed restores Kelvin's dispersion relation (??). The remaining task is simply to take differentiation of (5.6) with respect to ω_0 under the constraint of $D = 0$. By a repeated use of $D = 0$, we eventually arrive at a representation of the wave energy in a tidy form:

$$E^{(m)} = \frac{2\pi\omega_0}{\omega_0 - m} \left\{ 1 + \frac{(k_0/\eta_m)^2 K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}} \left[\frac{2(\omega_0 + m)}{\omega_0 - m} + \left(\frac{m(\omega_0 + m)}{2} + k_0^2 \right) \frac{K_{|m|}}{k_0 K_{|m|-1} + |m| K_{|m|}} \right] \right\} \left(A_0^{(m)}\right)^2. \tag{5.7}$$

With a view to gaining an insight into the result of §4, we sketch in figure 1 the wave energy $E^{(+)}$, divided by 2π, of the left-handed helical wave ($m = 1$) as a function of k_0. The wave energy of $m = -1$ is exactly the same at the same value of k_0 for the same order of branch, as is evident from the invariance property of (5.7) with respect to a replacement $m = -m$, $\omega_0 \to -\omega_0$. The energy of the primary mode is drawn with a solid thick line, and that of the first three branches of retrograde higher radial modes ($|\omega_0| < 1$), counted as $n = -1, -2, -3$, and of the first three branches of cograde higher radial modes ($|\omega_0| > 1$), counted as $n = 1, 2, 3$, are drawn with solid and dashed lines respectively. Here the minus sign is tentatively used in the counter of branch for the sake of convenience. The energy of the primary mode is positive in the entire wavenumber range. It starts from zero at $k_0 = 0$ and increases monotonically with k_0. The energy of a cograde higher mode as well is positive, but it increases without bound as $k_0 \to 0$ and monotonically decreases with k_0. The energy of a retrograde higher mode is negative for k_0 smaller than the value at which the dispersion curve transversally crosses the axis of $\omega_0 = 0$. It becomes negative infinity in the limit of $k_0 \to 0$, monotonically increases with k_0, and changes its sign at the intersection point. This singular behaviour is reflective of confinement of wave amplitude in the core.

The wave energy for bending waves, calculated from Cairns' formula, exhibits no contradiction with the spectra calculated in §4. The resonance instability of non-rotating waves ($\omega_0 = 0$), the dominant one, manifests itself at a degenerate eigenvalue with multiplicity two whose eigenfunction has zero energy. The resonance instability of rotating waves ($\omega_0 \neq 0$) stems from an eigenvalue collision either between the primary mode and a negative-energy retrograde higher mode or between a positive-energy and a negative-energy retrograde higher modes.

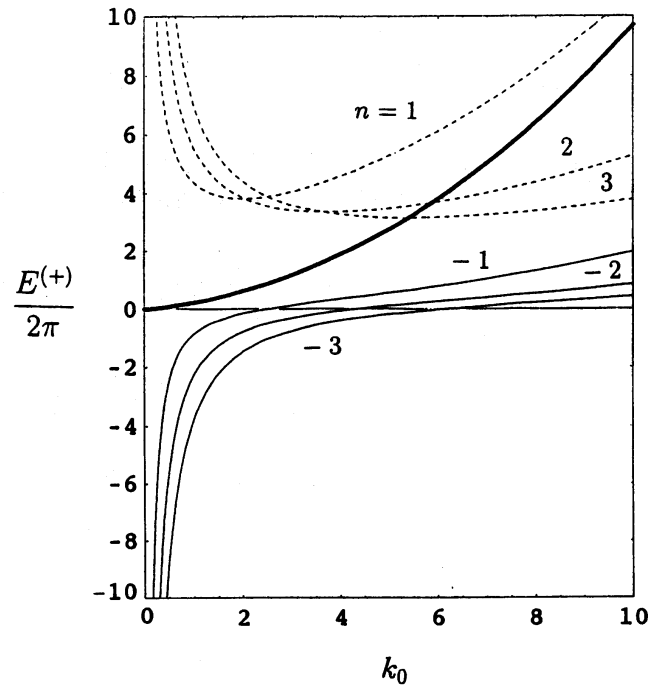

Figure 1: The wave energy $E^{(+)}$, normalized by 2π, of the the bending wave with $m = 1$ as functions of k_0, as given by (5.7). The solid thick line corresponds to the primary mode, solid lines to the retrograde higher radial modes, and dashed lines to the cograde higher radial modes.

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